

THE UNIVERSITY OF CHICAGO

ON CERTAIN SURFACES RELATED
COVARIANTLY TO A GIVEN
RULED SURFACE

A DISSERTATION

SUBMITTED TO THE FACULTY
OF THE OGDEN GRADUATE SCHOOL OF SCIENCE
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
DEPARTMENT OF MATHEMATICS

BY

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— AUGUST, 1923 —

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ON CERTAIN SURFACES RELATED COVARIANTLY TO A GIVEN RULED SURFACE

BY PROFESSOR N. B. MACLEAN

University of Manitoba, Winnipeg

1. INTRODUCTION

As the basis for the theory of ruled surfaces Wilczynski* has used a system of two ordinary linear homogeneous differential equations

$$(A) \quad \begin{aligned} y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z &= 0, \\ z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z &= 0, \end{aligned}$$

with coefficients p_{ik}, q_{ik} functions of an independent variable x .

To each *fundamental system*† of simultaneous solutions y_i, z_i ($i=1 \dots 4$), of (A), there corresponds a pair of directrix curves C_y, C_z of the ruled surface, generated respectively by the points P_y, P_z whose homogeneous space coordinates are (y_1, y_2, y_3, y_4) and (z_1, z_2, z_3, z_4) .

By suitable transformations‡ system (A) may be reduced to another of the same form

$$(F) \quad \begin{aligned} y'' + p_{12}z' + q_{11}y + q_{12}z &= 0, \\ z'' + p_{21}y' + q_{21}y + q_{22}z &= 0, \end{aligned}$$

in which $p_{11} = p_{22} = 0$; $p'_{12} = 2q_{12}$, $p'_{21} = 2q_{21}$; $u_{12} = u_{21} = 0$.§ For system (F), which is the flecnodal canonical form of (A), P_y and P_z are the *flecnodal points* of the generator g joining them, and C_y, C_z the two branches of the *flecnodal curve* of S .||

The *complex points*¶ of g will be denoted by $P_{\bar{y}}$ and $P_{\bar{z}}$ and the two branches of the *complex curve* of S by $C_{\bar{y}}$ and $C_{\bar{z}}$.

The osculating planes at P_y, P_z of g to the two branches C_y, C_z of the flecnodal curve, and those at $P_{\bar{y}}, P_{\bar{z}}$ of g to the two branches $C_{\bar{y}}, C_{\bar{z}}$ of the complex

*Wilczynski: Projective Differential Geometry of Curves and Ruled Surfaces, p. 126 *et seq.* This text hereafter referred to as W.

†W. p. 127. In this paper D is assumed $\neq 0$; the integrating ruled surface S is not a developable, W. p. 130.

‡W. pp. 116, 117.

§W. p. 96 (20).

||Cayley, Mathematical Papers, vol. II, p. 29; W. p. 149. Moreover, θ_4, p_{12}, p_{21} , are all assumed $\neq 0$, so that the two branches of the flecnodal curve are distinct, and neither C_y nor C_z a straight line; W. p. 150.

¶W. p. 206.

curve of S , form a tetrahedron, which will be found to be non-degenerate in general. This tetrahedron will be referred to as the *flecnode-complex tetrahedron* of the integrating ruled surface S . As x varies, the edges of this tetrahedron will generate six new ruled surfaces, while the vertices will describe space curves, each lying on three of the new ruled surfaces. The line of intersection of the osculating planes to the flecnode curve will be denoted by g_F , and the ruled surface it generates by S_F ; for the complex curve, the line of intersection of the osculating planes will be denoted by g_C , and the corresponding ruled surface by S_C .

It is the purpose in this paper to make a study of these six new ruled surfaces, associated covariantly with the original ruled surface S , and to find the differential equations of form (A) defining them. Special attention will be directed to a consideration of the surfaces S_F and S_C and of their relations to one another and to the original ruled surface; in particular, a necessary and sufficient condition for coincidence will be derived.

The methods and notations of Wilczynski will be used throughout. These have become familiar to mathematicians through his many contributions. The writer wishes here to acknowledge gratefully his indebtedness to Professor Wilczynski for his helpful suggestions and encouragement in the preparation of this paper.

2. THE DIFFERENTIAL EQUATIONS OF THE RULED SURFACES— S , S_F AND S_C

The Surface S .

If the surface S is referred to $C_{\bar{y}}$, $C_{\bar{z}}$ as directrix curves, the differential equations of form (A) in $\bar{y}$, $\bar{z}$ where $\bar{y}$ and $\bar{z}$ are given by*

$$(1) \quad \bar{y} = \sqrt{p_{21}} y + \sqrt{p_{12}} z; \quad \bar{z} = \sqrt{p_{21}} y - \sqrt{p_{12}} z;$$

assume the form†

$$(2) \quad \bar{y}'' + \bar{p}_{11}\bar{y}' + \bar{p}_{12}\bar{z}' + \bar{q}_{11}\bar{y} + \bar{q}_{12}\bar{z} = 0,$$

$$\bar{z}'' + \bar{p}_{21}\bar{y}' + \bar{p}_{22}\bar{z}' + \bar{q}_{21}\bar{y} + \bar{q}_{22}\bar{z} = 0,$$

where

$$\bar{p}_{11} = \sqrt{p_{12} p_{21}} - \left(\frac{q_{12}}{p_{12}} + \frac{q_{21}}{p_{21}} \right), \quad \bar{p}_{22} = -\sqrt{p_{12} p_{21}} - \left(\frac{q_{12}}{p_{12}} + \frac{q_{21}}{p_{21}} \right),$$

$$\bar{p}_{12} = \bar{p}_{21} = \left(\frac{q_{12}}{p_{12}} - \frac{q_{21}}{p_{21}} \right),$$

(3)

$$\bar{q}_{11} = \bar{q}_{22} = \frac{1}{2} \left(\frac{\Delta_2}{p_{21}^2} + \frac{\Delta_1}{p_{12}^2} \right) = \frac{1}{2} (\delta_2 + \delta_1),$$

$$\bar{q}_{12} = \bar{q}_{21} = \frac{1}{2} \left(\frac{\Delta_2}{p_{21}^2} - \frac{\Delta_1}{p_{12}^2} \right) = \frac{1}{2} (\delta_2 - \delta_1). \ddagger$$

*W, p. 207.

†A. F. Carpenter, Trans. Am. Math. Soc., vol. 16, 1915 (p. 517): hereafter referred to as A.F.C.

‡For the values of Δ_1 , Δ_2 see W. p. 230 (6), in which $p_{11} = p_{22} = 0$, $p'_{12} = 2q_{12}$, $p'_{21} = 2q_{21}$.

The Surface S_F .

The equations of the osculating planes to the flecnode curves C_y , C_z at the points P_y and P_z of the generator g of S are, respectively,

$$(4) \quad |x y y' y''| = 0, \text{ and } |x z z' z''| = 0.*$$

The differential equations of form (A) for the surface S_F in homogeneous plane coordinates u , v may be found as follows:

Denote by u_1, u_2, u_3, u_4 and by v_1, v_2, v_3, v_4 the cofactors of x_1, x_2, x_3, x_4 in the two equations of (4), respectively. Omitting subscripts, these may be written $u = |yy'y''|$, $v = |zz'z''|$. Then u and v may be taken as the plane coordinates of the two planes of (4) which intersect in g_F . To find the differential equations of form (A) satisfied by u and v , we eliminate y and z and their derivatives from u and v and from the derivatives of u and v . Making use of the canonical form (F), it is found that

$$(5) \quad \begin{aligned} u &= -p_{12}|yy'z'| - q_{12}|yy'z|, \\ v &= -p_{21}|zz'y'| - q_{21}|zz'y|, \end{aligned}$$

$$(6) \quad \begin{aligned} u' &= -3q_{12}|yy'z'| + (p_{12}q_{22} - q'_{12})|yy'z|, \\ v' &= -3q_{21}|zz'y'| + (p_{21}q_{11} - q'_{21})|zz'y|, \end{aligned}$$

$$(7) \quad \begin{aligned} u'' &= (p_{12}q_{22} - 4q'_{12})|yy'z'| + (3q_{12}q_{22} + p'_{12}q_{22} + p_{12}q'_{22} - q''_{12})|yy'z| + \Delta_1|zz'y|, \\ v'' &= (p_{21}q_{11} - 4q'_{21})|zz'y'| + (3q_{21}q_{11} + p'_{21}q_{11} + p_{21}q'_{11} - q''_{21})|zz'y| + \Delta_2|yy'z|. \end{aligned}$$

Employing (5) and (6) to eliminate $|yy'z'|$, $|yy'z|$, $|zz'y'|$ and $|zz'y|$ from (7), the system of equations satisfied by u and v may be written as

$$(8) \quad \begin{aligned} u'' + a_{11}u' + a_{12}v' + b_{11}u + b_{12}v &= 0, \\ v'' + a_{21}u' + a_{22}v' + b_{21}u + b_{22}v &= 0, \end{aligned}$$

where

$$(9) \quad \begin{aligned} a_{11} &= \frac{d}{dx} \log \frac{1}{\Delta_1}, \quad a_{12} = -p_{21} \frac{\Delta_1}{\Delta_2}, \quad a_{21} = -p_{12} \frac{\Delta_2}{\Delta_1}, \quad a_{22} = \frac{d}{dx} \log \frac{1}{\Delta_2}, \\ b_{11} &= q_{22} - 4 \frac{q'_{12}}{p_{12}} + 3 \frac{q_{12}}{p_{12}} \frac{d}{dx} \log \Delta_1, \quad b_{12} = 3q_{21} \frac{\Delta_1}{\Delta_2}, \\ b_{21} &= 3q_{12} \frac{\Delta_2}{\Delta_1}, \quad b_{22} = q_{11} - 4 \frac{q'_{21}}{p_{21}} + 3 \frac{q_{21}}{p_{21}} \frac{d}{dx} \log \Delta_2. \end{aligned}$$

* $|xyy'y''|$ is here written for $\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ y'_1 & y'_2 & y'_3 & y'_4 \\ y''_1 & y''_2 & y''_3 & y''_4 \end{vmatrix}$.

The system of differential equations of form (A) for the surface S_F in homogeneous point coordinates may be found in the following manner:

An arbitrary point on the tangent to C_y at P_y of g is given by $y' + \lambda y$, where λ is an arbitrary function of x . This will be the point, P_r , where the tangent to C_y at P_y meets g_F when λ is chosen so that $y' + \lambda y$ lies on the osculating plane to C_z at P_z of g . That is, $|y' + \lambda y, z, z', z''|$ must vanish. Using the second equation of (F), this reduces to $(\lambda p_{21} - q_{21})D = 0$. As p_{21} , D are assumed in §1 to be different from zero, $\lambda = \frac{q_{21}}{p_{21}}$, and P_r is the point $\frac{q_{21}}{p_{21}} y + y'$ or, written homogeneously, $q_{21}y + p_{21}y'$.

Similarly, the tangent to C_z at P_z of g meets the generator g_F in the point $q_{12}z + p_{12}z'$, which we call P_s . The curves C_r , C_s described by P_r , P_s on S_F are now taken as directrix curves, and, writing,

$$(10) \quad r = \lambda y + y', \quad s = \mu z + z', \quad \text{where } \lambda = \frac{q_{21}}{p_{21}}, \mu = \frac{q_{12}}{p_{12}},$$

we obtain, on successive differentiation,

$$(11) \quad r' - \lambda r + p_{12}s = -\delta_2 y,$$

$$s' + p_{21}r - \mu s = -\delta_1 z,$$

$$(12) \quad r'' - \lambda r' + p_{12}s' - \lambda' r + p'_{12}s = -\delta_2 y' - \delta'_2 y,$$

$$s'' + p_{21}r' - \mu s' + p'_{21}r - \mu' s = -\delta_1 z' - \delta'_1 z.$$

From (12) y, z, y', z' may be eliminated by the use of (10) and (11), and there results the system

$$(13) \quad r'' + c_{11}r' + c_{12}s' + d_{11}r + d_{12}s = 0$$

$$s'' + c_{21}r' + c_{22}s' + d_{21}r + d_{22}s = 0,$$

where

$$(14) \quad c_{11} = \frac{d}{dx} \log \frac{1}{\delta_2}, \quad c_{12} = p_{12}, \quad c_{21} = p_{21}, \quad c_{22} = \frac{d}{dx} \log \frac{1}{\delta_1},$$

$$d_{11} = q_{11} - 2\lambda' + \lambda \frac{d}{dx} \log \delta_2, \quad d_{12} = p'_{12} + \lambda p_{12} - p_{12} \frac{d}{dx} \log \delta_2,$$

$$d_{21} = p'_{21} + \mu p_{21} - p_{21} \frac{d}{dx} \log \delta_1, \quad d_{22} = q_{22} - 2\mu' + \mu \frac{d}{dx} \log \delta_1.*$$

The ruled surface S_F will be a developable if and only if the determinant $D(r'_k, s'_k, r_k, s_k)$, ($k=1 \dots 4$), vanishes.† Substitute the values of r and s , as given in (10), in $|r' s' r s|$, and the latter reduces to $\delta_1 \delta_2 D$. As D , p_{12} , p_{21} are assumed $\neq 0$, S_F will be a developable if and only if either $\Delta_1 = 0$ or $\Delta_2 = 0$. But if $\Delta_1 = 0$, then C_y is a plane curve, while if $\Delta_2 = 0$, C_z is a plane curve.‡

*For the values of δ_1, δ_2 cf. equations (3), section 2.

†W. p. 130.

‡W. p. 230.

Hence we have the result:

The ruled surface S_F will be a developable if and only if either C_y or C_z is a plane curve.

The Surface S_C .

An arbitrary point on the tangent to $C_{\bar{y}}$ at $P_{\bar{y}}$ of g is given by $\bar{\lambda} \bar{y} + \bar{y}'$. This will be the point $P_{\bar{r}}$, where the tangent to $C_{\bar{y}}$ intersects g_C , provided $\bar{\lambda} \bar{y} + \bar{y}'$ is on the osculating plane to $C_{\bar{z}}$ at $P_{\bar{z}}$ of g . Accordingly, we must have

$$|\bar{\lambda} \bar{y} + \bar{y}' \quad \bar{z} \quad \bar{z}' \quad \bar{z}''| = 0.$$

Making use of the second equation of (2), this condition reduces to

$$4(\bar{\lambda} \bar{p}_{21} - \bar{q}_{21}) \bar{p}_{12} \bar{p}_{21} |y' z' y z| = 0, \text{ and } \bar{\lambda} = \frac{\bar{q}_{21}}{\bar{p}_{21}}.$$

The point $P_{\bar{r}}$ is thus given by $\bar{r} = \bar{q}_{21} \bar{y} + \bar{p}_{21} \bar{y}'$, and, in like manner, the point $P_{\bar{s}}$, where the tangent to $C_{\bar{z}}$ at $P_{\bar{z}}$ meets the generator g_C of S_C is given by $\bar{s} = \bar{q}_{12} \bar{z} + \bar{p}_{12} \bar{z}'$.

The differential equations of form (A) for the ruled surface S_C , in which $C_{\bar{r}}$ and $C_{\bar{s}}$ are the directrix curves, are found as follows:

Writing

$$(15) \quad \bar{r} = \bar{\lambda} \bar{y} + \bar{y}', \quad \bar{s} = \bar{\lambda} \bar{z} + \bar{z}', \quad \text{where } \bar{\lambda} = \frac{\bar{q}_{12}}{\bar{p}_{12}} = \frac{\bar{q}_{21}}{\bar{p}_{21}}, \text{ [cf. (3)],}$$

differentiating, and using (2) and (3), it is found that

$$(16) \quad \bar{r}' + (\bar{p}_{11} - \bar{\lambda}) \bar{r}' + \bar{p}_{12} \bar{s}' = (\bar{\lambda}' - \bar{q}_{11} + \bar{\lambda} \bar{p}_{11} - \bar{\lambda}^2) \bar{y} = q_1 \bar{y}, \text{ where } q_1 = \bar{\lambda}' - \bar{q}_{11} + \bar{\lambda} \bar{p}_{11} - \bar{\lambda}^2,$$

$$\bar{s}' + \bar{p}_{21} \bar{r}' + (\bar{p}_{22} - \bar{\lambda}) \bar{s}' = (\bar{\lambda}' - \bar{q}_{22} + \bar{\lambda} \bar{p}_{22} - \bar{\lambda}^2) \bar{z} = q_2 \bar{z}, \text{ where } q_2 = \bar{\lambda}' - \bar{q}_{22} + \bar{\lambda} \bar{p}_{22} - \bar{\lambda}^2,$$

while

$$(17) \quad \bar{r}'' + (\bar{p}_{11} - \bar{\lambda}) \bar{r}' + \bar{p}_{12} \bar{s}' + (\bar{p}'_{11} - \bar{\lambda}') \bar{r} + \bar{p}'_{12} \bar{s} = q_1 \bar{y}' + q'_1 \bar{y},$$

$$\bar{s}'' + \bar{p}_{21} \bar{r}' + (\bar{p}_{22} - \bar{\lambda}) \bar{s}' + \bar{p}'_{21} \bar{r} + (\bar{p}'_{22} - \bar{\lambda}') \bar{s} = q_2 \bar{z}' + q'_2 \bar{z}.$$

Using (15), (16) to eliminate $\bar{y}$, $\bar{z}$, $\bar{y}'$, $\bar{z}'$, from (17) the desired system of differential equations for S_C takes the form

$$(18) \quad \bar{r}'' + m_{11} \bar{r}' + m_{12} \bar{s}' + n_{11} \bar{r} + n_{12} \bar{s} = 0,$$

$$\bar{s}'' + m_{21} \bar{r}' + m_{22} \bar{s}' + n_{21} \bar{r} + n_{22} \bar{s} = 0,$$

where

$$(19) \quad m_{11} = \bar{p}_{11} - \frac{d}{dx} \log q_1, \quad m_{12} = \bar{p}_{12}, \quad m_{21} = \bar{p}_{21}, \quad m_{22} = \bar{p}_{22} - \frac{d}{dx} \log q_2,$$

$$n_{11} = \bar{q}_{11} - \bar{\lambda}' + q_1 \left(\frac{\bar{p}_{11} - \bar{\lambda}}{q_1} \right)', \quad n_{12} = \bar{q}_{12} + q_1 \left(\frac{\bar{p}_{12}}{q_1} \right)',$$

$$n_{21} = \bar{q}_{21} + q_2 \left(\frac{\bar{p}_{21}}{q_2} \right)', \quad n_{22} = \bar{q}_{22} - \bar{\lambda}' + q_2 \left(\frac{\bar{p}_{22} - \bar{\lambda}}{q_2} \right)',$$

and where, moreover,

$$\left(\frac{\bar{p}_{11} - \bar{\lambda}}{q_1} \right)' = \frac{d}{dx} \left(\frac{\bar{p}_{11} - \bar{\lambda}}{q_1} \right), \text{ etc.}$$

The surface S_C , characterized by equations (18), will be a developable if and only if the determinant $|\bar{r}' \bar{s}' \bar{r} \bar{s}| = 0$. By the use of (15) this reduces to the condition, $q_1 q_2 |\bar{y}' \bar{z}' \bar{y} \bar{z}| = 0$. Substituting the values of $\bar{y}$ and $\bar{z}$ in terms of y and z , as given in (1), this condition becomes $4p_{12}p_{21}q_1q_2|y'z'yz| = 0$. It has been assumed, however, that p_{12} , p_{21} and $|y'z'yz|$ are all different from zero, and hence $|\bar{r}' \bar{s}' \bar{r} \bar{s}|$ vanishes only when either q_1 or q_2 vanishes.

Hence, the ruled surface S_C will be a developable if and only if either q_1 or q_2 is zero.

3. THE FLECNODE-COMPLEX TETRAHEDRON. THE ASSOCIATED RULED SURFACES

For convenience of reference, the following formulae are set down:

$$(20) \quad \bar{y} = \sqrt{p_{21}} y + \sqrt{p_{12}} z, \quad \bar{z} = \sqrt{p_{21}} y - \sqrt{p_{12}} z, \text{ cf. (1);}$$

$$(21) \quad r = q_{21} y + p_{21} y', \quad s = q_{12} z + p_{12} z', \text{ cf. (10);}$$

$$(22) \quad \bar{r} = \bar{q}_{21} \bar{y} + \bar{p}_{21} \bar{y}', \quad \bar{s} = \bar{q}_{12} \bar{z} + \bar{p}_{12} \bar{z}', \text{ cf. (15);}$$

wherein, by (3), it is noted that $\bar{p}_{12} = \bar{p}_{21}$, and $\bar{q}_{12} = \bar{q}_{21}$.

The point, $2\sqrt{p_{21}} \bar{q}_{21} y + 2 \frac{\bar{p}_{21}}{\sqrt{p_{21}}} r$, is evidently on the tangent to C_y at P_y .

Using the value of r from (21), $2\sqrt{p_{21}} \bar{q}_{21} y + 2 \frac{\bar{p}_{21}}{\sqrt{p_{21}}} r$ becomes $2\sqrt{p_{21}} \bar{q}_{21} y + 2 \frac{\bar{p}_{21}}{\sqrt{p_{21}}} (q_{21} y + p_{21} y')$. From (20), $\bar{y} + \bar{z} = 2\sqrt{p_{21}} y$, while $\bar{y}' + \bar{z}' = \frac{2}{\sqrt{p_{21}}} (q_{21} y + p_{21} y')$. Thus the point, $2\sqrt{p_{21}} \bar{q}_{21} y + 2 \frac{\bar{p}_{21}}{\sqrt{p_{21}}} r$ is actually the point given by

$\bar{q}_{21}(\bar{y} + \bar{z}) + \bar{p}_{21}(\bar{y}' + \bar{z}')$, which, from (22), is evidently the point $\bar{r} + \bar{s}$. Similarly, $\bar{r} - \bar{s}$ is found to be a point on the tangent to C_z at P_z of g ; and $P_{\bar{r} + \bar{s}}$ and $P_{\bar{r} - \bar{s}}$ are points on the generator g_C .

Again, it may be proved that $\bar{y}'$, a point on the tangent to C_y at P_y of g , is a point on g_F . For, from (20), on differentiation, $\bar{y}' = \frac{1}{\sqrt{p_{21}}} (q_{21} y + p_{21} y') + \frac{1}{\sqrt{p_{12}}} (q_{12} z + p_{12} z')$, which is the point $\frac{1}{\sqrt{p_{21}}} r + \frac{1}{\sqrt{p_{12}}} s$, and, accordingly, a point on g_F . Similarly, it may be shown that $\bar{z}'$ is the point $\frac{1}{\sqrt{p_{21}}} r - \frac{1}{\sqrt{p_{12}}} s$, also on g_F .

Accordingly, we have the important result:

The tangents to the flecnodal curve at P_y, P_z of g , and the tangents to the complex curve at P_y^-, P_z^- of g , intersect not only g_F and g_C respectively, but also both pairs intersect g_F and g_C .

Denote by A, B, C, D the vertices of the flecnodal-complex tetrahedron, where ABC, ABD are the osculating planes to C_y, C_z at P_y, P_z respectively. The edges AB, CD of the flecnodal-complex tetrahedron are the generators g_F and g_C of the ruled surfaces S_F and S_C respectively. A is the point of intersection of g_F with ACD ; the osculating plane ACD contains the tangent $P_y P_y'$, and P_y' has been proved to be a point on g_F , hence the point A is the point P_y' . In like manner, B is the point P_z' , and, for analogous reasons, C and D are the points $P_{\bar{r}+\bar{s}}$ and $P_{\bar{r}-\bar{s}}$, respectively.

Equations (13) characterize the ruled surface S_F , where C_r, C_s are used as directrix curves, and equations (18) the ruled surface S_C with $C_{\bar{r}}, C_{\bar{s}}$ as directrix curves. We propose now to use as directrix curves those generated by the vertices A, B, C, D of the flecnodal-complex tetrahedron, and to find the differential equations of form (A) defining the six ruled surfaces generated by the six edges of the tetrahedron $ABCD$.

As the computation involved in obtaining these systems is tedious, an outline will be given in the case of one of them, and the results alone set down for the remainder.

For brevity, denote the points A, B, C and D by $P_\alpha, P_\beta, P_\gamma$ and P_δ respectively, and let us find the differential equations of form (A) for the ruled surface generated by the edge AC .

We may write

$$\begin{aligned} \alpha &= \bar{y}', \\ (23) \quad \gamma &= \bar{r} + \bar{s} = \bar{\lambda}(\bar{y} + \bar{z}) + \bar{y}' + \bar{z}', \text{ where, as in (15), } \bar{\lambda} = \frac{\bar{q}_{12}}{\bar{p}_{12}} = \frac{\bar{q}_{21}}{\bar{p}_{21}}. \end{aligned}$$

Differentiating, and making use of (2), (3), (14), and (16) it is found that

$$\begin{aligned} (24) \quad \alpha' &+ (\bar{p}_{11} - \bar{p}_{12})\alpha + \bar{p}_{12}\gamma = -\delta_1\bar{y}, \\ \gamma' &+ (\bar{p}_{11} - \bar{p}_{22})\alpha + (\bar{p}_{12} + \bar{p}_{22} - \bar{\lambda})\gamma = q_2(\bar{y} + \bar{z}). \end{aligned}$$

A second differentiation gives

$$\begin{aligned} (25) \quad \alpha'' &+ (\bar{p}_{11} - \bar{p}_{12})\alpha' + \bar{p}_{12}\gamma' + (p_{11}' - \bar{p}_{12}')\alpha + \bar{p}_{12}'\gamma = -\delta_1'\bar{y} - \delta_1\bar{y}', \\ \gamma'' &+ (\bar{p}_{11} - \bar{p}_{22})\alpha' + (\bar{p}_{12} + \bar{p}_{22} - \bar{\lambda})\gamma' + (\bar{p}_{11}' - \bar{p}_{22}')\alpha + (\bar{p}_{12}' + \bar{p}_{22}' - \bar{\lambda}')\gamma \\ &= q_2'(\bar{y} + \bar{z}) + q_2(\bar{y}' + \bar{z}'). \end{aligned}$$

From (25) $\bar{y}, \bar{z}, \bar{y}', \bar{z}'$, may be eliminated by the use of (23) and (24), and the resulting system for α and γ takes the form

$$\begin{aligned}
(26) \quad & \alpha'' + (\bar{p}_{11} - \bar{p}_{12} + c_{22})\alpha' + \bar{p}_{12}\gamma' + \left\{ \delta_1 \left(\frac{\bar{p}_{11} - \bar{p}_{12}}{\delta_1} \right)' + \delta_1 \right\} \alpha + \delta_1 \left(\frac{\bar{p}_{12}}{\delta_1} \right)' \gamma = 0, \\
& \gamma'' + (\bar{p}_{11} - \bar{p}_{22})\alpha' + (m_{22} + \bar{p}_{12})\gamma' + \left\{ q_2 \left(\frac{\bar{p}_{11} - \bar{p}_{22}}{q_2} \right)' + \bar{\lambda}(\bar{p}_{11} - \bar{p}_{22}) \right\} \alpha \\
& \quad + \left\{ q_2 \left(\frac{\bar{p}_{22} + \bar{p}_{12} - \bar{\lambda}}{q_2} \right)' - q_2 + \bar{\lambda}(\bar{p}_{12} + \bar{p}_{22} - \bar{\lambda}) \right\} \gamma = 0,
\end{aligned}$$

where, as formerly, $\left(\frac{\bar{p}_{11} - \bar{p}_{12}}{\delta_1} \right)' = \frac{d}{dx} \left(\frac{\bar{p}_{11} - \bar{p}_{12}}{\delta_1} \right)$, etc.

In a similar manner, β and δ , α and δ , and β and γ satisfy systems (27), (28) and (29) respectively, as listed below:

$$\begin{aligned}
(27) \quad & \beta'' + (\bar{p}_{22} + \bar{p}_{21} + c_{11})\beta' + \bar{p}_{21}\delta' + \left\{ \delta_2 \left(\frac{\bar{p}_{22} + \bar{p}_{21}}{\delta_2} \right)' + \delta_2 \right\} \beta + \delta_2 \left(\frac{\bar{p}_{21}}{\delta_2} \right)' \delta = 0, \\
& \delta'' + (\bar{p}_{11} - \bar{p}_{22})\beta' + (m_{11} - \bar{p}_{21})\delta' + \left\{ q_1 \left(\frac{\bar{p}_{11} - \bar{p}_{22}}{q_1} \right)' + \bar{\lambda}(\bar{p}_{11} - \bar{p}_{22}) \right\} \beta \\
& \quad + \left\{ q_1 \left(\frac{\bar{p}_{11} - \bar{p}_{21} - \bar{\lambda}}{q_1} \right)' - q_1 + \bar{\lambda}(\bar{p}_{11} - \bar{p}_{21} - \bar{\lambda}) \right\} \delta = 0,
\end{aligned}$$

$$\begin{aligned}
(28) \quad & \alpha'' + (\bar{p}_{11} + \bar{p}_{12} + c_{11})\alpha' - \bar{p}_{12}\delta' + \left\{ \delta_2 \left(\frac{\bar{p}_{11} + \bar{p}_{12}}{\delta_2} \right)' + \delta_2 \right\} \alpha - \delta_2 \left(\frac{\bar{p}_{12}}{\delta_2} \right)' \delta = 0, \\
& \delta'' + (\bar{p}_{11} - \bar{p}_{22})\alpha' + (m_{22} - \bar{p}_{12})\delta' + \left\{ q_2 \left(\frac{\bar{p}_{11} - \bar{p}_{22}}{q_2} \right)' + \bar{\lambda}(\bar{p}_{11} - \bar{p}_{22}) \right\} \alpha \\
& \quad + \left\{ q_2 \left(\frac{\bar{p}_{22} - \bar{p}_{12} - \bar{\lambda}}{q_2} \right)' - q_2 + \bar{\lambda}(\bar{p}_{22} - \bar{p}_{12} - \bar{\lambda}) \right\} \delta = 0,
\end{aligned}$$

$$\begin{aligned}
(29) \quad & \beta'' + (\bar{p}_{22} - \bar{p}_{21} + c_{22})\beta' + \bar{p}_{21}\gamma' + \left\{ \delta_1 \left(\frac{\bar{p}_{22} - \bar{p}_{21}}{\delta_1} \right)' + \delta_1 \right\} \beta + \delta_1 \left(\frac{\bar{p}_{21}}{\delta_1} \right)' \gamma = 0, \\
& \gamma'' + (\bar{p}_{22} - \bar{p}_{11})\beta' + (m_{11} + \bar{p}_{21})\gamma' + \left\{ q_1 \left(\frac{\bar{p}_{22} - \bar{p}_{11}}{q_1} \right)' + \bar{\lambda}(\bar{p}_{22} - \bar{p}_{11}) \right\} \beta \\
& \quad + \left\{ q_1 \left(\frac{\bar{p}_{11} + \bar{p}_{21} - \bar{\lambda}}{q_1} \right)' - q_1 + \bar{\lambda}(\bar{p}_{11} + \bar{p}_{21} - \bar{\lambda}) \right\} \gamma = 0.
\end{aligned}$$

The differential equations of form (A) for S_F and S_C where C_r , C_s and $C_{\bar{r}}$, $C_{\bar{s}}$ were used respectively as directrix curves are given already by (13) and (18).

If, however, the curves described by P_α , P_β and by P_γ , P_δ are used as directrix curves for S_F , S_C respectively, the differential equations of form (A) for these two ruled surfaces assume forms (30) and (31);

$$(30) \quad \begin{aligned} \alpha'' + \left\{ \bar{p}_{11} + \frac{1}{2}(c_{11} + c_{22}) \right\} \alpha' + \left\{ \bar{p}_{12} + \frac{1}{2}(c_{11} - c_{22}) \right\} \beta' + \left\{ \bar{q}_{11} + \bar{p}'_{11} + \frac{1}{2}\bar{p}_{11}(c_{11} + c_{22}) \right. \\ \left. + \frac{1}{2}\bar{p}_{21}(c_{11} - c_{22}) \right\} \alpha + \left\{ \bar{q}_{12} + \bar{p}'_{12} + \frac{1}{2}\bar{p}_{12}(c_{11} + c_{22}) + \frac{1}{2}\bar{p}_{22}(c_{11} - c_{22}) \right\} \beta = 0, \\ \beta'' + \left\{ \bar{p}_{21} + \frac{1}{2}(c_{11} - c_{22}) \right\} \alpha' + \left\{ \bar{p}_{22} + \frac{1}{2}(c_{11} + c_{22}) \right\} \beta' + \left\{ \bar{q}_{21} + \bar{p}'_{21} + \frac{1}{2}\bar{p}_{11}(c_{11} - c_{22}) \right. \\ \left. + \frac{1}{2}\bar{p}_{21}(c_{11} + c_{22}) \right\} \alpha + \left\{ \bar{q}_{22} + \bar{p}'_{22} + \frac{1}{2}\bar{p}_{12}(c_{11} - c_{22}) + \frac{1}{2}\bar{p}_{22}(c_{11} + c_{22}) \right\} \beta = 0, \end{aligned}$$

$$(31) \quad \begin{aligned} \gamma'' + \left\{ \frac{1}{2}(m_{11} + m_{22}) + m_{12} \right\} \gamma' + \frac{1}{2}(m_{11} - m_{22}) \delta' + \frac{1}{2}(n_{11} + n_{12} + n_{21} + n_{22}) \gamma \\ + \frac{1}{2}(n_{11} - n_{12} + n_{21} - n_{22}) \delta = 0, \\ \delta'' + \frac{1}{2}(m_{11} - m_{22}) \gamma' + \left\{ \frac{1}{2}(m_{11} + m_{22}) - m_{21} \right\} \delta' + \frac{1}{2}(n_{11} + n_{12} - n_{21} - n_{22}) \gamma \\ + \frac{1}{2}(n_{11} - n_{12} - n_{21} + n_{22}) \delta = 0. \end{aligned}$$

4. A CERTAIN ONE-PARAMETER FAMILY OF CURVES ON S . RECIPROCAL CONSIDERATIONS FOR THE SURFACE S_F . THE CONJUGATE FAMILY ON S , AND THE CORRESPONDING LAPLACE TRANSFORMATION

An arbitrary point P of the generator g of S is given by $\alpha y + \beta z$, where the ratio $\alpha : \beta$ is arbitrary. The plane of P and g_F determines a tangent, t , to S at P . Thus, one obtains a one-parameter family of curves on S such that the tangents to them along g intersect g_F . We proceed to investigate this family of curves.

The plane determined by P and g_F is given by the equation

$$(32) \quad |x, \alpha y + \beta z, q_{21}y + p_{21}y', q_{12}z + p_{12}z'| = 0.$$

As the independent variable changes, P describes a curve on S and $\frac{d}{dx}(\alpha y + \beta z)$, which is a point on the tangent, t , to this curve at P , will lie on the plane (32), provided

$$(33) \quad \left| \frac{d}{dx}(\alpha y + \beta z), \alpha y + \beta z, q_{21}y + p_{21}y', q_{12}z + p_{12}z' \right| = 0.$$

On expanding, simplifying, and dividing out by $D = |y'z'yz| \neq 0$, this condition reduces to

$$(34) \quad \frac{\alpha'}{\alpha} - \frac{\beta'}{\beta} = \frac{1}{2} \left(\frac{p'_{21}}{p_{21}} - \frac{p'_{12}}{p_{12}} \right).$$

The integration of (34) gives

$$(35) \quad \frac{\alpha}{\beta} = c \sqrt{\frac{p_{21}}{p_{12}}}, \text{ where } c \text{ is an arbitrary constant.}$$

It may be noted that for $c = \pm 1$, the complex points, $P_{\bar{y}}$, $P_{\bar{z}}$, of g are obtained.

This important result may be stated as follows:

The one-parameter family of curves on S , characterized by the property that the tangents to them along the generator g intersect g_F , are those determined by the condition of forming a constant cross-ratio with the complex curves of S .

We may now obtain the interpretation of the point on t given by $\frac{d}{dx}(ay+\beta z)$, where $\alpha : \beta$ satisfies (35). Take $\alpha = c\sqrt{p_{21}}$, $\beta = \sqrt{p_{12}}$, then

$$\frac{d}{dx}(ay+\beta z) = \frac{c}{\sqrt{p_{21}}}(q_{21}y+p_{21}y') + \frac{1}{\sqrt{p_{12}}}(q_{12}z+p_{12}z') = \frac{a}{p_{21}}r + \frac{\beta}{p_{12}}s,$$

which is clearly the point of intersection of t with g_F .

Again, suppose $\alpha : \beta$ satisfies the relation (35); the point on the tangent t , given by $2\bar{q}_{12}(ay+\beta z) + 2\bar{p}_{12}\frac{d}{dx}(ay+\beta z)$, reduces, by the use of (20) and (22), to $c(\bar{r}+\bar{s}) + (\bar{r}-\bar{s})$, which is seen to be a point on the generator g_C .

Thus, we have deduced the further result:

The one-parameter family of curves on S , determined by (35), have the property that the tangents to them along g , intersect not only g_F but also g_C .

Similarly a point P' of g_F and g determine a tangent t' to S_F at P' . Thus, in a reciprocal manner, is obtained a one-parameter family of curves on S_F such that the tangents to them along g_F intersect the generator g of S .

In this case, let P' of g_F be given by $\gamma r + \delta s$. We proceed to find the ratio $\gamma : \delta$ so that this tangent t' may intersect g .

An arbitrary point on t' is given by

$$(36) \quad \frac{d}{dx}(\gamma r + \delta s) + \lambda(\gamma r + \delta s).$$

In order that t' may intersect g , the point given by (36) must, for a proper choice of λ , be a point of g , and its coordinates expressible therefore in terms of y and z alone. If, in (36), are substituted the values of r and s in terms of y, z, y', z' , and the conditions imposed that the resulting expression shall contain only terms in y and z , then γ and δ will be found to satisfy the equation

$$(37) \quad \frac{\gamma'}{\gamma} - \frac{\delta'}{\delta} + p_{21}\frac{\gamma}{\delta} - p_{12}\frac{\delta}{\gamma} - \frac{3}{2}\left(\frac{p'_{12}}{p_{12}} - \frac{p'_{21}}{p_{21}}\right) = 0.$$

In (37) replace $\frac{\gamma}{\delta}$ by $\left(\frac{p_{12}}{p_{21}}\right)^{\frac{3}{2}}\pi$, and there results the Riccati equation in π ,

$$(38) \quad \pi' + p_{12}\sqrt{\frac{p_{12}}{p_{21}}}\pi^2 = p_{21}\sqrt{\frac{p_{21}}{p_{12}}}.$$

From the characteristic property of any four solutions of a Riccati equation we deduce that:

Any four points of the generator g_F , which are such that the tangents to the curves they describe (at the points where these curves meet g_F) intersect the generator g , have the property that their anharmonic ratio is constant.

The Laplace Transformation.

The one-parameter family of curves on S , characterized by the property that the tangents to them along g intersect g_F , and, as we have proved, g_C also, has associated with it a conjugate family, which we proceed to find.

An arbitrary curve of the above family is that described by P_t where $t = \bar{y} + c\bar{z}$, c being an arbitrary constant. In this investigation the complex curves are taken as directrix curves for S , and hence the system of differential equations used is that given by (2).

The two asymptotic directions at P_t are given by the generator g , and by the line joining P_t to P_a where $a = 2y' + p_{11}\bar{y} + p_{12}\bar{z} + c(2z' + p_{21}y + p_{22}\bar{z})$.^{*} If P_a is joined to P_t' this line will intersect g in some point P_u . To find u , it may be noted that the coordinates of an arbitrary point on the line P_aP_t' is given by $a + \lambda t'$. In order that this may be the point P_u , $a + \lambda t'$ must be expressible in terms of $\bar{y}$ and $\bar{z}$ alone; that is

$$(39) \quad 2\bar{y}' + \bar{p}_{11}\bar{y} + \bar{p}_{12}\bar{z} + c(2\bar{z}' + \bar{p}_{21}\bar{y} + \bar{p}_{22}\bar{z}) + \lambda(\bar{y}' + c\bar{z}') = S\bar{y} + R\bar{z}.$$

From this it follows that $\lambda = -2$, and

$$(40) \quad u = (\bar{p}_{11} + c\bar{p}_{21})\bar{y} + (\bar{p}_{12} + c\bar{p}_{22})\bar{z}.$$

Thus $a = 2\bar{y}' + \bar{p}_{11}\bar{y} + \bar{p}_{12}\bar{z} + c(2\bar{z}' + \bar{p}_{21}\bar{y} + \bar{p}_{22}\bar{z}) = u + 2t'$, and v , the harmonic conjugate of t' with respect to u and a , is found to be equal to $u + t'$. The direction conjugate to P_tP_v is thus given by the line P_tP_v .

If the surface S is referred to C_t , C_u as directrix curves, a system of differential equations of form (A) in t and u may be found. One of these is sufficient for present purposes.

We have

$$(41) \quad \begin{aligned} t &= \bar{y} + c\bar{z}, \\ u &= (\bar{p}_{11} + c\bar{p}_{21})\bar{y} + (\bar{p}_{12} + c\bar{p}_{22})\bar{z}. \end{aligned}$$

On differentiating we obtain,

$$(42) \quad \begin{aligned} t' &= \bar{y}' + c\bar{z}', \\ u' &= (\bar{p}_{11} + c\bar{p}_{21})\bar{y}' + (\bar{p}_{12} + c\bar{p}_{22})\bar{z}' + (\bar{p}'_{11} + c\bar{p}'_{21})\bar{y} + (\bar{p}'_{12} + c\bar{p}'_{22})\bar{z}, \end{aligned}$$

and,

$$(43) \quad t'' = \bar{y}'' + c\bar{z}'' = -(\bar{p}_{11} + c\bar{p}_{21})\bar{y}' - (\bar{p}_{12} + c\bar{p}_{22})\bar{z}' - (\bar{q}_{11} + c\bar{q}_{21})\bar{y} - (\bar{q}_{12} + c\bar{q}_{22})\bar{z}.$$

This latter equation may be written, using the second from (42),

$$(44) \quad t'' + u' = (\bar{p}'_{11} + c\bar{p}'_{21} - \bar{q}_{11} - c\bar{q}_{21})\bar{y} + (\bar{p}'_{12} + c\bar{p}'_{22} - \bar{q}_{12} - c\bar{q}_{22})\bar{z}.$$

The use of (41) enables $\bar{y}$ and $\bar{z}$ to be eliminated from (44), and one of the two equations of form (A), satisfied by t and u , takes the form

$$(45) \quad t'' + u' + s_{11}t + s_{12}u = 0,$$

^{*}W. p. 146.

where

$$(46) \quad \begin{aligned} s_{11} &= \frac{-(\bar{p}'_{11} + c\bar{p}'_{21} - \bar{q}_{11} - c\bar{q}_{21})(\bar{p}_{12} + c\bar{p}_{22}) + (\bar{p}'_{12} + c\bar{p}'_{22} - \bar{q}_{12} - c\bar{q}_{22})(\bar{p}_{11} + c\bar{p}_{21})}{\bar{p}_{12} + c\bar{p}_{22} - c(\bar{p}_{11} + c\bar{p}_{21})}, \\ s_{12} &= \frac{c(\bar{p}'_{11} + c\bar{p}'_{21} - \bar{q}_{11} - c\bar{q}_{21}) - (\bar{p}'_{12} + c\bar{p}'_{22} - \bar{q}_{12} - c\bar{q}_{22})}{\bar{p}_{12} + c\bar{p}_{22} - c(\bar{p}_{11} + c\bar{p}_{21})}. \end{aligned}$$

The tangent $P_t P_v$, as x varies, will now generate a developable surface, a Laplace transform of S , and we proceed to find a point on the edge of regression.

The coordinates of an arbitrary point P_ρ on $P_t P_v$ is given by $v + \lambda t$; this will be a point on the edge of regression if $\frac{d}{dx}(v + \lambda t)$ is again expressible in terms of t and v alone, that is, if

$$(47) \quad v' + \lambda t' + \lambda' t = Mt + Nv.$$

But $v = u + t'$, whence (47) becomes

$$(48) \quad u' + t'' + \lambda t' + \lambda' t = Mt + Nv,$$

and, by the use of (45),

$$(49) \quad -s_{11}t - s_{12}u + \lambda(v - u) + \lambda' t = Mt + Nv,$$

whence $\lambda = -s_{12}$, and $\rho = v - s_{12}t$, and the locus of P_ρ gives the corresponding Laplace transform of S .

5. THE FLECNODE-COMPLEX QUADRIC. ITS RELATION TO THE OSCULATING HYPERBOLOID H OF S

It may be noted that the tangents along g to the one-parameter family of curves on S given in §4, equation (35), constitute the rulings of the second kind of the quadric determined by the three generators g , g_F , and g_C .

Let us use as tetrahedron of reference that determined by $P_y, P_z, P_\rho, P_\sigma$,* where $\rho = 2y' + p_{12}z$, $\sigma = 2z' + p_{21}y$, and define a point given by an expression of the form, $x_1y + x_2z + x_3\rho + x_4\sigma$, as having the coordinates x_1, x_2, x_3, x_4 .

As already noted, an arbitrary curve of the family cuts g in the point P_t , where $t = c\sqrt{p_{21}}y + \sqrt{p_{12}}z$. A general point on the quadric determined by g , g_F and g_C will then be given by $at + \beta t'$. Since

$$\begin{aligned} at + \beta t' &= a(c\sqrt{p_{21}}y + \sqrt{p_{12}}z) + \beta \left(c\sqrt{p_{21}}y' + \sqrt{p_{12}}z' + c\frac{q_{21}}{\sqrt{p_{21}}}y + \frac{q_{12}}{\sqrt{p_{12}}}z \right) \\ &= \left(ac\sqrt{p_{21}} + \beta c\frac{q_{21}}{\sqrt{p_{21}}} - \frac{\beta}{2}p_{21}\sqrt{p_{12}} \right) y + \left(a\sqrt{p_{12}} - \frac{\beta}{2}c p_{12}\sqrt{p_{21}} + \beta\frac{q_{12}}{\sqrt{p_{12}}} \right) z \\ &\quad + \frac{1}{2}\beta c\sqrt{p_{21}}\rho + \frac{1}{2}\beta\sqrt{p_{12}}\sigma, \end{aligned}$$

it follows that the homogeneous point-coordinates of an arbitrary point on the quadric determined by g , g_F , and g_C are

*W. p. 191.

$$(50) \quad \begin{aligned} x_1 &= ac\sqrt{p_{21}} + \beta c \frac{q_{21}}{\sqrt{p_{21}}} - \frac{1}{2}\beta p_{21}\sqrt{p_{12}}, & x_3 &= \frac{1}{2}\beta c\sqrt{p_{21}}, \\ x_2 &= a\sqrt{p_{12}} - \frac{1}{2}\beta c p_{12}\sqrt{p_{21}} + \beta \frac{q_{12}}{\sqrt{p_{12}}}, & x_4 &= \frac{1}{2}\beta\sqrt{p_{12}}. \end{aligned}$$

The elimination of α , β and c leads to the equation of the quadric determined by the three generators g , g_F and g_C in the form

$$(51) \quad x_1x_4 - x_2x_3 - \left\{ p_{12}x_3^2 - 2\left(\frac{q_{12}}{p_{12}} - \frac{q_{21}}{p_{21}}\right)x_3x_4 - p_{21}x_4^2 \right\} = 0,$$

which we shall call the *Flecnode-Complex Quadric** of the given integrating ruled surface S .

If from the quadratic covariants $C_1^\dagger$, $C_2^\dagger$ and $C_5^\S$ of weights 1, 2 and 5 respectively, we form the covariant,

$$K = -C_1 - \frac{\theta_9}{2\theta_{10}}C_2 + \frac{C_5}{\theta_4},$$

of degree 2 and weight 1, we find that K is expressible in the form, $y\sigma - z\rho + \bar{y}\bar{z} - 2\left(\frac{q_{12}}{p_{12}} - \frac{q_{21}}{p_{21}}\right)yz$. If, with each of the points $P_y, P_z, P_\rho, P_\sigma, P_{\bar{y}}, P_{\bar{z}}$ there is associated its polar plane with respect to the osculating quadric, $x_1x_4 - x_2x_3 = 0$, and if, from the equations of these polar planes, an expression is built up precisely as K is built up out of the coordinates of the points, this expression equated to zero gives exactly the equation of the *flecnode-complex quadric*.

The quadric (51) is tangent to the osculating quadric, $x_1x_4 - x_2x_3 = 0$, along g , so that g , counted twice, is a part of the intersection of the two quadrics. The two rulings of the second kind on the flecnode-complex quadric, which at the same time lie on the osculating quadric, and therefore comprise the rest of the intersection, pass through the two points of g given by

$$p_{12}p_{21}y + (p_{12}q_{21} - p_{21}q_{12} \pm \sqrt{(p_{12}q_{21} - p_{21}q_{12})^2 + p_{12}^3p_{21}^3})z.$$

A few results will now be pointed out which follow readily from the above considerations.

An arbitrary point on g_F is given by $\lambda r + \mu s$, which may be written, using (21), in the form,

$$(\lambda q_{21} - \frac{1}{2}\mu p_{12}p_{21})y + (\mu q_{12} - \frac{1}{2}\lambda p_{12}p_{21})z + \frac{1}{2}\lambda p_{21}\rho + \frac{1}{2}\mu p_{12}\sigma.$$

The homogeneous point-coordinates of an arbitrary point on g_F , referred to the local tetrahedron of reference are therefore,

$$x_1 = \lambda q_{21} - \frac{1}{2}\mu p_{12}p_{21}, \quad x_2 = \mu q_{12} - \frac{1}{2}\lambda p_{12}p_{21}, \quad x_3 = \frac{1}{2}\lambda p_{21}, \quad x_4 = \frac{1}{2}\mu p_{12}.$$

*A. F. Carpenter: Bull. Amer. Math. Soc., vol. XXVII, 1921, p. 3. The *Flecnode-Complex Quadric* is the quadric of the family K_θ , associated with the generator g of S , whose equation is invariant in form.

† W. p. 125. ‡ W. p. 125. § W. p. 208, formula (36). $||$ W. p. 191.

This will be a point on H if λ and μ satisfy the relation

$$(52) \quad \lambda^2 p_{21} - 2\lambda\mu \left(\frac{q_{12}}{p_{12}} - \frac{q_{21}}{p_{21}} \right) - \mu^2 p_{12} = 0.$$

The two values of $\lambda : \mu$, satisfying (52), when substituted in $\lambda r + \mu s$, determine the two points of intersection of g_F with H . These two points are harmonic conjugates with respect to P_r and P_s if $\frac{q_{12}}{p_{12}} - \frac{q_{21}}{p_{21}} = \frac{1}{2} \frac{p'_{12}}{p_{12}} - \frac{1}{2} \frac{p'_{21}}{p_{21}} = 0$, i.e., if

$\frac{p_{12}}{p_{21}} = \text{const.}$, that is, if S belongs to a linear complex.* The generator g_F will be a tangent to H if the two roots of (52) are equal, that is, if

$$(53) \quad (p_{12}q_{21} - p_{21}q_{12})^2 + p^3_{12}p^3_{21} = 0.†$$

The osculating planes to C_y and C_z at P_y and P_z of g are given by‡

$$\begin{aligned} p_{12}x_2 + p^2_{12}x_3 - 2q_{12}x_4 &= 0, \\ p_{21}x_1 - 2q_{21}x_3 + p^2_{21}x_4 &= 0, \text{ where again } u_{12} = u_{21} = 0. \end{aligned}$$

The null-points of these planes by means of the osculating linear complex, $p_{12}\omega_{13} + p_{21}\omega_{42} = 0$ §, are respectively $(-p_{12}p_{21}, -2q_{12}, 0, -p_{12})$ and $(2q_{12}, p_{12}p_{21}, p_{21}, 0)||$, and their line of junction intersects the hyperboloid H in two points. These latter form an harmonic group with the former if S belongs to a linear complex, while their line of junction is a tangent to H if (53) holds. Since $p_{11} = p_{22} = u_{12} = u_{21} = 0$, condition (53) may be written as $\theta^3_4\theta^2_9 + 16\theta^3_{10} = 0$.¶

6. THE OSCULATING PLANES TO THE COMPLEX CURVES. RELATIONS WITH RESPECT TO THE OSCULATING LINEAR COMPLEX

The osculating plane to $C_{\bar{y}}$ at $P_{\bar{y}}$ of g is given by $|x\bar{y}\bar{y}'\bar{y}''| = 0$. Using the first equation from (2), and (20), this equation becomes

$$(54) \quad |x\bar{y}\bar{y}'y'| + q|x\bar{y}\bar{y}'y| = 0, \text{ where } q = \frac{\bar{q}_{12}}{\bar{p}_{12}} + \frac{q_{21}}{p_{21}} = \frac{\bar{q}_{21}}{\bar{p}_{21}} + \frac{q_{12}}{p_{12}}.$$

From the substitution of the values of $\bar{y}$, $\bar{y}'$ in terms of y and z and of their derivatives, there results

$$(55) \quad p_{12}|xyz'y'| + \sqrt{p_{12}p_{21}}|xyy'z'| - qp_{12}|xyz'z'| - 4 \frac{q_{12}p_{21} - q_{21}p_{12} + qp_{12}p_{21}}{\sqrt{p_{12}p_{21}}} |xyz'y'| = 0.$$

But, the semi-covariants of system (F) , reduce to

$$\rho = 2y' + p_{12}z, \quad \sigma = 2z' + p_{21}y,$$

and, using these to determine y' , z' in terms of y , z , ρ and σ , (55) may then be written

$$(56) \quad \begin{aligned} & p_{12}|xz\rho\sigma| + \frac{1}{2}(\bar{p}_{11} - \bar{p}_{22})|xy\rho\sigma| - \{2qp_{12} + \frac{1}{2}p_{12}(\bar{p}_{11} - \bar{p}_{22})\}|xyz\sigma| \\ & - (\bar{p}_{11} - \bar{p}_{22})\{\bar{p}_{12} + q + \frac{1}{4}(\bar{p}_{11} - \bar{p}_{22})\}|xyz\rho| = 0. \end{aligned}$$

*W. p. 167. †W. p. 214. ‡W. p. 214. §W. p. 206. ||W. p. 215. ¶W. p. 214.

As the tetrahedron of reference is that determined by $P_y, P_z, P_\rho, P_\sigma$ the equation of the osculating plane to C_y at P_y of g thus becomes

$$(57) \quad \begin{aligned} & p_{12}x_1 - \frac{1}{2}(\bar{p}_{11} - \bar{p}_{22})x_2 - \left\{ 2qp_{12} + \frac{1}{2}p_{12}(\bar{p}_{11} - \bar{p}_{22}) \right\} x_3 \\ & + (\bar{p}_{11} - \bar{p}_{22}) \left\{ \bar{p}_{12} + q + \frac{1}{4}(\bar{p}_{11} - \bar{p}_{22}) \right\} x_4 = 0. \end{aligned}$$

In a similar manner, the equation of the osculating plane to C_z at P_z of g may be found; and, thus referred to the local tetrahedron of reference, the osculating planes to the two branches of the complex curve may be set down as

$$(58) \quad \begin{aligned} & 4p_{12}x_1 - 2(\bar{p}_{11} - \bar{p}_{22})x_2 - \left\{ 8p_{12}q + 2p_{12}(\bar{p}_{11} - \bar{p}_{22}) \right\} x_3 \\ & + (\bar{p}_{11} - \bar{p}_{22}) (4\bar{p}_{12} + 4q + \bar{p}_{11} - \bar{p}_{22})x_4 = 0, \\ & 4\bar{p}_{12}x_1 + 2(\bar{p}_{11} - \bar{p}_{22})x_2 - \left\{ 8\bar{p}_{12}q - 2p_{12}(\bar{p}_{11} - \bar{p}_{22}) \right\} x_3 \\ & - (\bar{p}_{11} - \bar{p}_{22}) (4\bar{p}_{12} + 4q - \bar{p}_{11} + \bar{p}_{22})x_4 = 0, \end{aligned}$$

where, as already noted, $q = \frac{q_{12}}{\bar{p}_{12}} + \frac{q_{21}}{\bar{p}_{21}} = \frac{\bar{q}_{21}}{\bar{p}_{21}} + \frac{q_{21}}{\bar{p}_{21}}$, and where, moreover, $u_{12} = u_{21} = 0$.

Carpenter* found that, when the flecnode curves were non-rectilinear, distinct, plane-curves, so also were the two branches of the complex curve. A necessary and sufficient condition for C_y and C_z to be plane curves is $\Delta_1 = \Delta_2 = 0$.† This condition enables equations (58) to be reduced to those given by him.‡

Suppose, however, we assume $\delta_1 = \delta_2$. From (3) it follows that $\bar{q}_{12} = \bar{q}_{21} = 0$, and, accordingly, q in (58) reduces to $\frac{q_{21}}{\bar{p}_{21}}$; and, since $\bar{p}_{11} - \bar{p}_{22} = 2\sqrt{p_{12}p_{21}}$, equations (58) become

$$(59) \quad \begin{aligned} & \frac{1}{\sqrt{p_{21}}}x_1 - \frac{1}{\sqrt{p_{12}}}x_2 - \left(\sqrt{p_{12}} + \frac{p'_{21}}{p_{21}\sqrt{p_{21}}} \right) x_3 + \left(\sqrt{p_{21}} + \frac{p'_{12}}{p_{12}\sqrt{p_{12}}} \right) x_4 = 0, \\ & \frac{1}{\sqrt{p_{21}}}x_1 + \frac{1}{\sqrt{p_{12}}}x_2 + \left(\sqrt{p_{12}} - \frac{p'_{21}}{p_{21}\sqrt{p_{21}}} \right) x_3 + \left(\sqrt{p_{21}} - \frac{p'_{12}}{p_{12}\sqrt{p_{12}}} \right) x_4 = 0, \end{aligned}$$

the same as those given by Carpenter.§

If the first equation of (59) is subtracted from the second and the result multiplied by $\frac{1}{2}(p_{12})^{\frac{3}{2}}$, and again the two equations of (59) added and the result multiplied by $\frac{1}{2}(p_{21})^{\frac{3}{2}}$, there result, respectively,

$$(60) \quad \begin{aligned} & p_{12}x_2 + p_{12}^2x_3 - p'_{12}x_4 = 0, \\ & p_{21}x_1 - p'_{21}x_3 + p_{21}^2x_4 = 0, \end{aligned}$$

which are the osculating planes to the two branches of the flecnode curve of S ,|

*A.F.C. pp. 518, 519.

†W. p. 230.

‡A.F.C. p. 519, equations (36) and (37).

§A.F.C. p. 519.

||W. p. 214.

and we have thus seen that the two planes given by (59), and the two planes given by (60), have a common line of intersection.

Combining the results obtained above, with the harmonic property of the four points $P_x, P_y, P_z, P_{\bar{z}}$ of the generator g , we may state in a more general form the results Carpenter* obtained (when the flecnode curves and therefore the complex curves were plane curves), as follows:

If the two branches of the flecnode curve and the two branches of the complex curve are distinct, and if the integrating ruled surface S is subject to the condition, $\delta_1 - \delta_2 = 0$, then the four osculating planes have a common line of intersection. Moreover, these four planes form an harmonic pencil of planes, in which the former two are separated harmonically by the latter two.

The osculating linear complex, $p_{12}\omega_{13} + p_{21}\omega_{42} = 0$,† written in the expanded form,

$$(61) \quad -p_{12}x_3y_1 + p_{21}x_4y_2 + p_{12}x_1y_3 - p_{21}x_2y_4 = 0,$$

gives the point-plane correspondence,

$$(62) \quad u_1 = -p_{12}x_3, \quad u_2 = p_{21}x_4, \quad u_3 = p_{12}x_1, \quad u_4 = -p_{21}x_3;$$

and the plane-point correspondence,

$$(63) \quad x_1 = p_{21}u_3, \quad x_2 = -p_{12}u_4, \quad x_3 = -p_{21}u_1, \quad x_4 = p_{12}u_2.$$

The null-points of the two osculating planes to the complex curve given by (58), as determined by the osculating linear complex, (61), are, therefore, respectively,

$$(64) \quad (2p_{21}\{4q + \bar{p}_{11} - \bar{p}_{22}\}, \{\bar{p}_{11} - \bar{p}_{22}\}\{4\bar{p}_{12} + 4q + \bar{p}_{11} - \bar{p}_{22}\}, 4p_{21}, 2\{\bar{p}_{11} - \bar{p}_{22}\}), \\ (2p_{21}\{4q - \bar{p}_{11} + \bar{p}_{22}\}, -\{\bar{p}_{11} - \bar{p}_{22}\}\{4\bar{p}_{12} + 4q - \bar{p}_{11} + \bar{p}_{22}\}, 4p_{21}, -2\{\bar{p}_{11} - \bar{p}_{22}\}).$$

Denote these two points of (64) by P_ν and $P_{\nu'}$. An arbitrary point on their line of junction is $\lambda\nu + \mu\nu'$. This line intersects the osculating hyperboloid H in the two points resulting from the values of $\lambda : \mu$ obtained from

$$(65) \quad \bar{p}_{12}\lambda^2 + (\bar{p}_{11} - \bar{p}_{22})\lambda\mu - \bar{p}_{12}\mu^2 = 0.$$

These points of intersection will coincide, that is, the line joining the null-points of the two osculating planes to the complex curve will be tangential to H , if $(\bar{p}_{11} - \bar{p}_{22})^2 + 4\bar{p}_{12}^2 = 0$, which condition may be expressed as $\theta^3_4\theta^2_9 + 16\theta^3_{10} = 0$. This interesting result may be thus stated:

If $\theta^3_4\theta^2_9 + 16\theta^3_{10} = 0$, the ruled surface S has the following characteristic property. The planes, which osculate the complex curve at the two points of its intersection with any generator, intersect in a line which is tangent to the osculating hyperboloid.

It is verified easily that the coordinates of the two points, given by (64), satisfy equation (51), so that we may add the following theorem:

The ruled surface S has the following characteristic property. The null-points of the planes, which osculate the complex curve at the two points of its intersection with any generator, as determined by the osculating linear complex, lie on the flecnode-complex quadric of S .

*A.F.C. p. 520.

†W. p. 206.

7. A NECESSARY AND SUFFICIENT CONDITION FOR THE FLECNODE-COMPLEX
TETRAHEDRON TO BECOME DEGENERATE. THE RESULTING INVARIANT
RELATION

The generator g_F intersects the generator g_C if the four points $P_r, P_s, P_{\bar{r}}$ and $P_{\bar{s}}$ are coplanar. If this is so, the determinant $|r^{(k)}, s^{(k)}, \bar{r}^{(k)}, \bar{s}^{(k)}|$, ($k=1 \dots 4$), must vanish. By referring to equations (21) and (22), it is seen that this condition may be expressed as

$$(66) \quad \left| \frac{q_{21}}{p_{21}} y + y', \frac{q_{12}}{p_{12}} z + z', \frac{\bar{q}_{21}}{\bar{p}_{21}} \bar{y} + \bar{y}', \frac{\bar{q}_{12}}{\bar{p}_{12}} \bar{z} + \bar{z}' \right| = 0.$$

By the use of relations (20) $\bar{y}, \bar{z}, \bar{y}', \bar{z}'$ are expressed in terms of y, z, y' and z' ; when, in this manner, $\bar{y}, \bar{z}, \bar{y}'$, and $\bar{z}'$, are eliminated from (66), and the resulting equation simplified, the condition for the intersection of g_F and g_C reduces to

$$(67) \quad \sqrt{p_{12}p_{21}} \cdot \frac{\bar{q}_{12}}{\bar{p}_{12}} \cdot \frac{\bar{q}_{21}}{\bar{p}_{21}} |yzy'z'| = 0.$$

As p_{12}, p_{21} and $D = |yzy'z'|$, are all assumed to be different from zero, it follows that

$$(68) \quad \bar{q}_{12} = \bar{q}_{21} = 0,$$

which, by the use of (3), becomes

$$(69) \quad \delta_1 - \delta_2 = 0.$$

That this necessary condition for the intersection of g_F and g_C is also sufficient is at once evident.

It may be proved, moreover, that, if $\delta_1 - \delta_2 = 0$, then g_F and g_C coincide. For, an arbitrary point on g_C is given by $\lambda\bar{r} + \mu\bar{s}$. From (22), if $\delta_1 - \delta_2 = 0$, that is, if $\bar{q}_{12} = \bar{q}_{21} = 0$, it follows that $\lambda\bar{r} + \mu\bar{s}$ is expressible in terms of $\bar{y}'$ and $\bar{z}'$ alone. In §3 it was proved that $\bar{y}' = \frac{1}{\sqrt{p_{21}}}r + \frac{1}{\sqrt{p_{12}}}s$ and $\bar{z}' = \frac{1}{\sqrt{p_{21}}}r - \frac{1}{\sqrt{p_{12}}}s$, and accordingly $\lambda\bar{r} + \mu\bar{s}$ becomes expressible in terms of r and s alone. Thus, if $\delta_1 - \delta_2 = 0$, an arbitrary point on g_C is a point of g_F . If necessary, it might be shown conversely that an arbitrary point on g_F is a point of g_C . Hence, if $\delta_1 - \delta_2 = 0$, g_F and g_C coincide.

Condition (69) may be written, using the relations given in (3), in the form

$$(70) \quad \frac{p_{21}}{p_{12}} \Delta_1 - \frac{p_{12}}{p_{21}} \Delta_2 = 0.*$$

Carpenter† found that

$$\delta_1 - \delta_2 = \frac{8}{3} \theta_4 \sqrt{\theta_4} (6\theta_4\theta_{10}^2 - \theta_4\theta_{20} - \theta_{10}\theta_{14}),$$

*Bulletin Amer. Math. Soc., vol. 28, No. 6, 1922, p. 283. Notice of paper by Professor A. F. Carpenter.

†A.F.C. p. 513.

and, as θ_4 has been assumed to be different from zero, the condition expressed in (69) may be written in the form of the invariant relation,

$$(71) \quad 6\theta_4\theta_{10}^2 - \theta_4\theta_{20} - \theta_{10}\theta_{14} = 0.$$

Combining the results here obtained with those of section 6, we have the general theorem:

A necessary and sufficient condition, that the planes osculating the two branches of the flecnode curve at the two points where the curve intersects any generator g , may have a common line of intersection with the planes osculating the two branches of the complex curve where it meets the same generator, is that the invariant relation $6\theta_4\theta_{10}^2 - \theta_4\theta_{20} - \theta_{10}\theta_{14} = 0$, shall hold. Moreover, if this condition is satisfied, then these four planes form a harmonic pencil of planes, in which the former two are separated harmonically by the latter two.

The writer has considered also certain questions which arise naturally in connection with the invariants of the systems of differential equations of form (A) defining the ruled surfaces S_F and S_C . In particular, when the integrating ruled surface, S , is subject to the condition $6\theta_4\theta_{10}^2 - \theta_4\theta_{20} - \theta_{10}\theta_{14} = 0$, in which case S_F and S_C coincide, the possibility of S and S_F having the same relative invariants has been examined, and some preliminary results obtained. He hopes to be able to extend these considerations and develop more general properties.

